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TRANSIENT THERMAL BEHAVIOUR OF GROUND ELECTRODES
PART II: CYLINDRICAL ELECTRODES

G. J. Berg,* and A. G. Herron.**

SUMMARY

Transient thermal performance of a ground electrode arrangement consisting of concentric cylinders with axis in the earth's surface has been studied. The solutions obtained involve integrals which must be evaluated numerically. Some numerical results are presented, assuming typical moist soil conditions. A procedure is suggested whereby the transient thermal performance of single cylinder arrangements may be predicted.

1. *Introduction*

In Part I, spherical electrode arrangements were considered and solutions for the transient temperature rise following a step rise in electrode potential were presented. [1] While the spherical electrode is of interest for a number of reasons, a cylindrical configuration corresponds much more closely to practical electrode arrangements.

The analysis of the transient thermal behaviour of a cylindrical electrode system, which is the subject of this paper, has been based on assumptions similar to those for the spheres. The chosen model consists of an inner cylinder of radius a and an outer concentric cylinder of radius b located with the axis in the earth's surface. Radial symmetry is assumed for both electric and thermal fields, implying that the cylinders are long compared to their radii and that end effects can be neglected. These assumptions do not permit increasing b without limit in order to study the single cylinder electrode. But an approach is suggestes by which b is related to the length of the electrically equivalent single cylinder. An indication of the thermal performance of the single cylinder arrangement is thus obtained.

Assumptions and boundary conditions are as follows:

1. A step rise in electric potential from 0 to U_0 occurs at the inner cylinder at the time $t = 0$.
2. The outer cylinder remains at zero potential.
3. The inner cylinder has negligible heat capacity compared to the near environment, hence temperature gradient at radius a remains zero.
4. The outer cylinder is held at constant temperature, T_{amb}.
5. Initially the temperature is uniform and equal to ambient throughout the medium.
6. The medium between the cylinders is homogeneous and isotropic with temperature-independent physical parameters.

It will be noted that the assumption of radial symmetry excludes heat loss to the atmosphere. This assumption makes the boundary conditions for the electric and thermal fields similar so that it becomes possible to express final temperature rise of the electrode in terms of the electrode potential and physical parameters of the surrounding medium. [2].

* Department of Electrical Engineering, The University of Calgary, Canada
** Calgary Power Limited, Calgary, Canada

2. *Analysis*

Under the conditions assumed, temperature will be a function of radius and time only. Consider a unit length of the cylinder arrangement. The thermal field is governed by the Diffusion Equation with a heat generation term [3],

$$\nabla^2 T + \rho_T\ (\bar{E}.\bar{i}) = \frac{1}{k}\frac{\partial T}{\partial t} \tag{1}$$

where $T = T(r,t)$, [°C], ρ_T = thermal resistivity, [m°C/W], $\bar{E}$ = electric field gradient [V/m], $\bar{i}$ = current density [A/m], and k = thermal diffusivity [W^2/sec].

The heat generation term may now be expressed as

$$\rho_T\ E_r \cdot i_r = \frac{\rho_T}{\rho_E}\ (-\ \text{grad}\ U)^2 = \frac{G_c}{r^2} \tag{2}$$

where ρ_E = electrical resistivity, [Ωm], and G_c is the heat generation constant. The potential distribution between the cylinders becomes

$$U = U(r) = U_o \frac{\ln \frac{r}{b}}{\ln \frac{a}{b}},\quad a \leq r \leq b \tag{3}$$

and hence the heat generation constant for the cylindrical arrangement is found to be

$$G_c = \frac{\rho_T}{\rho_E}\ \frac{U_o^2}{(\ln\frac{b}{a})^2} \tag{4}$$

Combining Eqs. (4) and (2) with (1), the basic partial differential equation for the temperature field is obtained,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{G_c}{r^2} = \frac{1}{k}\frac{\partial T}{\partial t} \tag{5}$$

Solution may be assumed to have the form

$$T(r,t) = R_c(r)\ S_c(t) + f_c(r) \tag{6}$$

Since the time dependent part must vanish as $t \to \infty$, $f_c(r)$ represents the final or steady state temperature distribution between the cylinders.

Substituting for T in eq. (5) yields

$$S_c\ \frac{\partial^2 R_c}{\partial R^2} + \frac{S_c}{r}\frac{\partial R_c}{\partial r} + \frac{\partial^2 f_c}{\partial r^2} + \frac{1}{r}\frac{\partial f_c}{\partial r} + \frac{G_c}{r^2} = \frac{R_c}{k}\frac{\partial S_c}{\partial t} \tag{7}$$

To satisfy the conditions of the problem, the sum of the purely radius dependent term in Eq. (7) must equal zero, i.e.,

$$\frac{\partial^2 f_c}{\partial r^2} + \frac{1}{r}\frac{\partial f_c}{\partial r} + \frac{G_e}{r^2} = 0 \tag{8}$$

Equation (8) thus concerns the steady state temperature distribution. The solution, with boundary conditions as assumed, is found as

$$f_c(r) = T_{amb} + \tfrac{1}{2} G_c \left[\ln^2 b - \ln^2 r - 2(\ln a) \ln \frac{b}{r}\right] \tag{9}$$

The maximum electrode temperature rise is obtained from Eq. (9) by taking $r = a$:

$$\Delta T_{max} = f_c(a) - T_{amb} = \tfrac{1}{2} G_c \ln^2 \frac{b}{a} \tag{10}$$

By substituting for G_c from Eq. (4), we find

$$\Delta T_{max} = \tfrac{1}{2} \frac{\rho_T}{\rho_E} U_o^2 \tag{11}$$

which is identical to the result for the spheres.

The remaining part of Eq. (7) is as follows

$$S_c \frac{\partial^2 R_c}{\partial r^2} + \frac{S_c}{r}\frac{\partial R_c}{\partial r} = \frac{R_c}{k}\frac{\partial S_c}{\partial t} \tag{12}$$

Following the Separation of Variables technique, let

$$\frac{1}{k S_c}\frac{\partial S_c}{\partial t} = -\lambda^2,$$

hence

$$S_c = S_c(t) = C_1 e^{-\lambda^2 kt} \tag{13}$$

The radius dependent function, R_c, is governed by

$$r^2 \frac{\partial^2 R_c}{\partial r^2} + \frac{\partial R_c}{\partial r} + (r\lambda)^2 R_c = 0 \tag{14}$$

which corresponds to Bessel's equation of order zero with (λr) as independent variable. The solution to Eq. (14) is therefore

$$R_c(r) = C_2 J_o(\lambda r) + C_3 Y_o(\lambda r) \tag{15}$$

Now, combining Eqs. (9), (13) and (15) and taking the sum of all possible solution, the temperature function can be expressed as

$$T(r,t) = \sum_{n=0}^{\infty} e^{-\lambda_n kt} \left\{ A_n J_o(\lambda_n r) + B_n Y_o(\lambda_n r) \right\} + f_c(r) \tag{16}$$

Zero temperature gradient at $r = a$ requires that

$$\sum_{n=0}^{\infty} e^{-\lambda_n kt} \lambda_n \left\{ A_n J_o'(\lambda_n r) + B_n Y_o'(\lambda_n r \right\}_{r=a} = 0 \tag{17}$$

i.e.

$$A_n J_o'(\lambda_n r) + B_n Y_o'(\lambda_n r)\Big/_{r=a} = 0 \tag{17}$$

At the outer boundary, the temperature remains constant, equal to ambient. This condition will be satisfied by Eq. (16) by requiring

$$A_n J_o(\lambda_n b) + B_n Y_o(\lambda_n b) = 0 \tag{18}$$

Combining Eqs. (17) and (18), a relation is shown to exist between the constants A_n and B_n:

$$\frac{A_n}{B_n} = -\frac{Y_o'(\lambda_n a)}{J_o'(\lambda_n a)} \tag{19}$$

Also, an equation which defines the eigenvalues is obtained,

$$J_o'(\lambda_n a)\ Y_o(\lambda_n b) - J_o(\lambda_n b)\ Y_o'(\lambda_n a) = 0 \tag{20}$$

Considering Eq. (19), it will be noted that A_n and B_n can be expressed in terms of a common constant, K_n, by letting

$$\left.\begin{aligned} A_n &= -K_n Y_o'(\lambda_n a) \\ B_n &= K_n J_o'(\lambda_n a) \end{aligned}\right\}.. \tag{21}$$

Hence the solution for the temperature function may be expressed as follows

$$T(r,t) = \sum_0^{\infty} e^{-\lambda_n kt} K_n W_n(r) + f_c(r) \tag{22}$$

where

$$W_n(r) = J_o'(\lambda_n a)\ Y_o(\lambda_n r) - J_o(\lambda_n r)\ Y_o'(\lambda_n a) \tag{23}$$

The initial condition must be employed in determining the constants K_n. For $t = 0$, Eq. (22) reduces to

$$\sum_0^{\infty} K_n W_n(r) = \tfrac{1}{2} G_c \left[\ln^2 r - \ln^2 b + 2(\ln a)\ln\frac{b}{r}\right] \tag{24}$$

Orthogonal properties of the function $W_n(r)$ over the interval $[a, b]$ permit evaluation of the constants K_n. It can be shown [4] that

$$\int_a^b r\, W_n(r)\ W_m(r)\ dr = H_n \delta_{nm}$$

$$\delta_{nm} \begin{cases} = 1 \text{ for } n = m \\ = 0 \text{ for } n \neq m \end{cases}$$

The orthogonality constant, H_n, is found as

$$H_n = \frac{1}{2\lambda_n^2}\left[b^2\left(\frac{\partial W_n(r)}{\partial r}\Big/_{r=b}\right)^2 - \lambda_n^2\, a^2\, W_n^2(a)\right] \quad (25)$$

By multiplying both sides of Eq. (24) by $rW_n(r)$ and integrating from a to b, we have

$$K_n H_n = \tfrac{1}{2}\, G_c \int_a^b r\, W_n(r)\left[\ln^2 r - \ln^2 b + 2(\ln a)\ln\frac{b}{r}\right] dr$$

Hence, using Eq. (23), the constants K_n may be found from the relation

$$K_n = \frac{G_c}{2H_n}\int_a^b \left\{ r\left[J_o'(\lambda_n a)\, Y_o(\lambda_n r) - J_o(\lambda_n r)\, Y_o'(\lambda_n a)\right]\left[\ln^2 r - \ln^2 b + 2(\ln a)\ln\frac{b}{r}\right]\right\} dr \quad (26)$$

3. *Results*

Some numerical results have been obtained, using the above solution and moist soil condition for the medium between the cylinders. Figure 1 shows the temperature rise at the inner cylinder surface as a function of time for various ratios $\frac{b}{a}$ and for a = 5 cm. The time required for the inner electrode to reach 63% of final temperature rise (apparent time constant) is shown in Figure 2, with a =5, 10 and 20 cm., and $\frac{b}{a}$ as independent variable.

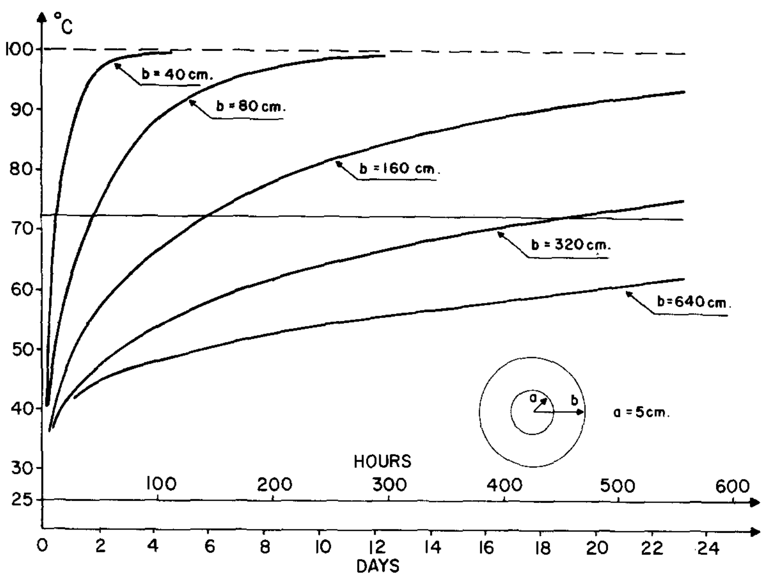

Fig. 1 Temperature rise of inner cylinder electrode

It will be noted that the initial rate of temperature rise at the electrode is relatively high. The rate then drops to a low value which tends to remain almost unchanged for a considerable time. Very similar features have been observed experimentally with buried cylindrical electrodes [5].

An important question now arises: How can the results for the concentric

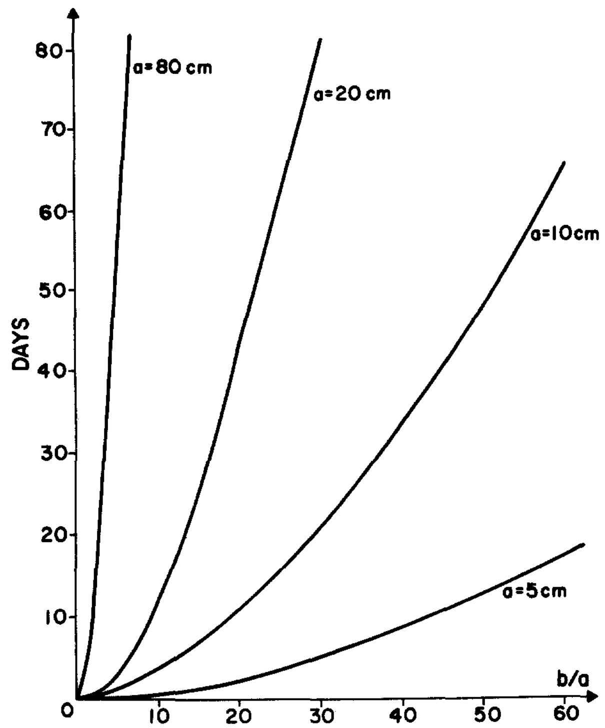

Fig. 2 Time required for inner cylinder electrode to reach 63% of final temperature rise

cylinder arrangement be used to predict thermal performance of a single-cylinder electrode?

As the ratio $\frac{b}{a}$ increases without limit, the resultant electrical resistance of the electrode arrangement also increases without limit.
In practice, we will have a cylindrical electrode of finite length and its resistance is finite for that reason. If the actual length of the straight single hemi-cylinder electrode is l, the radius is a, and l is much greater than a, one can show[6] that the electrical resistance becomes

$$R_{hc} \cong \frac{\rho_E}{\pi} \ln \frac{\frac{3}{4}l}{a} \quad \Omega/\text{unit length} \tag{27}$$

The corresponding expression for the infinitely long concentric hemicylinder arrangement is

$$R_{hcc} = \frac{\rho_E}{\pi} \ln \frac{b}{a} \quad \Omega/\text{unit length} \tag{28}$$

Thus, if end effects were excluded the two arrangements would have the same electrical resistance by choosing the outer cylinder radius $b = \frac{3}{4}l$. Closer examination indicates that the two arrangements exhibit very similar electric and thermal field characteristics near the (common) inner electrode under the conditions assumed. Hence it is suggested that for the purpose of studying the near field and transient temperature changes of a long single cylinder, (length l) one can use the concentric cylinder model studied above with $b = \frac{3}{4}l$.

As an example, consider a single horizontal cylinder of radius a = .05 m. and length l = 4.1 m. It will have a resistance-to-ground equal that of a sphere of radius .5 m. In accordance with the approach suggestes above, the corresponding concentric cylinder model has a ratio $\frac{b}{a} = 61$. Figure 2 shows that it will take about 17 days for the cylinder to reach 63% of final temperature rise. The corresponding time for the sphere of the same resistance-to-ground was 38 days [1].

It is of interest also to compare these results with the so-called Short-term Time Constant, τ_i, for the electrode systems considered. τ_i is defined as the ratio between final temperature rise and initial rate of rise, that is

$$\tau_i = \frac{\Delta T_{max}}{\lim\limits_{t\to 0} \left.\frac{\partial T}{\partial r}\right|_{r=a}} \begin{cases} = \frac{a^2}{2k} \text{ for sphere} \\ = \frac{a^2}{2k}(\ln \frac{b}{a})^2 \text{ for cyl.} \end{cases} \tag{29}$$

For the cylinder arrangement in the above example, τ_i = 0.51 days, while the electrically equivalent spherical electrode has a Short-term Time Constant of 3.05 days.

The analysis indicates quite clearly that the cylinder arrangement will heat up much faster than the spherical arrangement with identical electrical resistance-to-ground. In both cases, however, the results must be judged against the assumptions made at the outset. For instance, neglecting the heat loss to the atmosphere, and reduction in electrical resistivity of the soil as temperature increases, will cause some reduction in the maximum temperature reached by the electrode.

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